ON $\pi(x + y) \leqq \pi(x) + \pi(y)$

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In this note, as usual, P_i will denote the ith prime number ($P_1 = 2$), and $\pi(x)$ the number of prime numbers $\leqq x$. Tables of prime numbers strongly suggest the hypothesis that for all integers x, $y \geqq 2$

$$\pi(x + y) \leqq \pi(x) + \pi(y) \tag{1}$$

which is an old, though still unproved, conjecture. The best known results thus far may be summed up as follows:

(A) There exists a positive integer X, such that for all $x \geqq X$, $\pi(2x) < 2\pi(x)$ (Landau);

(B) There exists a positive constant A, such that $\lim \sup_{x\to\infty} \pi(x+y) - \pi(x) \leqq Ay/(\log y)$ (Hardy and Littlewood);

(C) (1) is true if either x or y is $\leqq 132$ (Schinzel and Sierpiński).

In this note we shall obtain an inequality condition involving only prime numbers which is equivalent to the assertion that (1) is true for all integers x, $y \geqq 2$. Denoting by Q a positive integer such that (1) is true for all x, y, such that $x+y \leqq Q$, we may use this equivalence to easily improve the value of Q beyond the value 265 implied by C. In fact, we shall show that we may take $Q = 101{,}081$.

Precisely, our object will be to prove the following two theorems:

THEOREM I. (1) *is true for all integers* x, $y \geqq 2$, *if and only if for all integers* $n \geqq 3$ *and all integers* q, $1 \leqq q \leqq (n-1)/2$,

$$P_n \geqq P_{n-q} + P_{q+1} - 1 \text{ is true.} \tag{2}$$

THEOREM II. *If* (1) *is false for some integer* $x+y$, *then the smallest such value of* $x+y$ *is the smallest value of* P_n *for which* (2) *is false.*

We first prove some lemmas.

LEMMA I. (1) *is false for some* $x \geqq 2$, $y \geqq 2$, *if and only if there exist integers* $M \geqq 2$, $K \geqq 2$, *such that*

$$\pi(M + K) = \pi(M) + \pi(K), \tag{3}$$

$$M + K + 1 \text{ is prime,} \tag{4}$$

$$M + 1 \text{ is composite.} \tag{5}$$

Proof. If such M and K exist, then $\pi(M+K+1) = \pi(M+K) + 1 = \pi(M) + \pi(K) + 1 = \pi(M+1) + \pi(K) + 1 > \pi(M+1) + \pi(K)$ in contradiction to (1).

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On the other hand, suppose no such M and K exist. We may then prove (1) by induction as follows.

For all integers $k\geqq 2$, $\pi(2+k)\leqq\pi(k)+1=\pi(k)+\pi(2)$.

Suppose that for all integers m, $2\leqq m\leqq M$ (M is here an arbitrary integer $\geqq 2$)

$$\pi(m+k)\leqq\pi(m)+\pi(k). \tag{6}$$

We need then to show that for all integers $k\geqq 2$, $\pi(M+k+1)\leqq\pi(M+1)+\pi(k)$. We consider three cases.

Case I. $M+k+1$ is composite. Then by (6), $\pi(M+k+1)=\pi(M+k)\leqq\pi(M)+\pi(k)\leqq\pi(M+1)+\pi(k)$.

Case II. $M+k+1$ is prime, $M+1$ also being prime. Then by (6), $\pi(M+k+1)=\pi(M+k)+1\leqq\pi(M)+\pi(k)+1=\pi(M+1)+\pi(k)$.

Case III. $M+k+1$ is prime, while $M+1$ is composite. Then by hypothesis, $\pi(M+k)\neq\pi(M)+\pi(k)$, and so it follows from (6), $\pi(M+k)\leqq\pi(M)+\pi(k)-1$. And hence $\pi(M+k+1)=\pi(M+k)+1\leqq\pi(M)+\pi(k)+1-1\leqq\pi(M+1)+\pi(k)$. And so our lemma is proved.

LEMMA II. (1) *is false for some* $x\geqq 2$, $y\geqq 2$ (*if and*) *only if there exist integers* $M_0\geqq 2$, $K_0\geqq 2$, *such that* M_0 *and* K_0 *satisfy* (3), (4), (5), *and also*

$$K_0+1 \text{ is prime.} \tag{7}$$

Proof. Suppose (1) is false for some $x\geqq 2$, $y\geqq 2$. Then by Lemma I there exist pairs of integers M, K, satisfying (3), (4), and (5). Let M_0 be the smallest value of M such that for some $K=K_0$, M_0 and K_0 satisfy (3), (4), and (5). Then it follows from the argument used in proving Lemma I that for all integers μ, $2\leqq\mu\leqq M_0$, and for all $k\geqq 2$

$$\pi(\mu+k)\leqq\pi(\mu)+\pi(k). \tag{8}$$

Since M_0 and K_0 satisfy (3) and (4) by hypothesis, taking $\mu=M_0$, $k=K_0+1$ in this inequality, we get

$$\begin{aligned}\pi(M_0)+\pi(K_0+1)\geqq\pi(M_0+K_0+1)&=\pi(M_0+K_0)+1\\&=\pi(M_0)+\pi(K_0)+1.\end{aligned}$$

So $\pi(K_0+1)\geqq\pi(K_0)+1$. But trivially, $\pi(K_0+1)\leqq\pi(K_0)+1$. Hence, $\pi(K_0+1)=\pi(K_0)+1$, that is K_0+1 is prime.

LEMMA III. *With* M_0 *and* K_0 *as in Lemma* II, $K_0\geqq M_0+2$.

Proof. Suppose first K_0 were $\leqq M_0$. Then from the definition of M_0, we may apply the inequality (8) with $\mu=K_0$, and $k=M_0+1$, and so $\pi(M_0+K_0+1)\leqq\pi(K_0)+\pi(M_0+1)$. But M_0 and K_0 satisfy (3), (4), and (5). Hence, $\pi(M_0+K_0+1)=\pi(M_0)+\pi(K_0)+1=\pi(M_0+1)+\pi(K_0)+1\geqq\pi(M_0+K_0+1)+1$, a contradiction.

Hence, $K_0\geqq M_0+1$. But K_0+1 is, by Lemma II, prime, and since $K_0\geqq 2$,

$K_0+1 \geqq 3$. Since M_0+K_0+1 is also prime, M_0 must be even. Since K_0 is also even, it follows that $K_0 \geqq M_0+2$.

LEMMA IV. (1) *is false for some integers* $x \geqq 2$, $y \geqq 2$, *if and only if there exists a prime number*, P_n, *and an integer* q, $1 \leqq q \leqq (n-1)/2$, *such that*

$$P_{n-q} + P_{q+1} - 3 \geqq P_n \geqq P_{n-q} + P_q + 1. \tag{9}$$

Proof. "If." Suppose such a P_n and q exist. Then in (1), take $x = P_n - P_{n-q} + 1$, $y = P_{n-q} - 1$. Then, $\pi(x+y) = \pi(P_n) = n = (n-q-1) + q + 1 = \pi(P_{n-q} - 1) + \pi(P_{q+1} - 1) + 1 > \pi(y) + \pi(P_{q+1} - 1) \geqq \pi(y) + \pi(P_n - P_{n-q} + 2) \geqq \pi(y) + \pi(x)$. So, $\pi(x+y) > \pi(x) + \pi(y)$.

"Only if." Suppose (1) is false for some $x \geqq 2$, $y \geqq 2$. Then by Lemmas I, II, and III, there exist integers $M_0 \geqq 2$, $K_0 \geqq 2$, such that, $\pi(M_0+K_0) = \pi(M_0) + \pi(K_0)$; M_0+K_0+1 is prime; K_0+1 is prime; M_0+1 is composite and odd; $K_0 \geqq M_0+2$.

Let $M_0+K_0+1 = P_n$, $K_0+1 = P_{n-q}$. Then, $\pi(P_n - 1) = \pi(M_0+K_0) = \pi(M_0) + \pi(K_0) = \pi(M_0) + \pi(P_{n-q} - 1)$ or $n-1 = \pi(M_0) + n - q - 1$, or $q = \pi(M_0)$. Hence, $P_q \leqq M_0 \leqq P_{q+1} - 1$. But M_0+1 is composite and odd. Hence, $P_q + 1 \leqq M_0 \leqq P_{q+1} - 3$. Consequently, by our choice of n and q, $P_{n-q} + P_q + 1 \leqq M_0 + K_0 + 1 \leqq P_{n-q} + P_{q+1} - 3$, or $P_{n-q} + P_q + 1 \leqq P_n \leqq P_{n-q} + P_{q+1} - 3$.

Finally, since $K_0 \geqq M_0+2$, $P_{n-q} - 1 \geqq M_0 + 2 \geqq P_q + 3$. Therefore, $P_{n-q} \geqq P_q + 4$, and so $n - q \geqq q + 1$, or $q \leqq (n-1)/2$. Only a simple additional argument remains to establish Theorem I, which we now restate.

THEOREM I. (1) *is true for all integers* x, $y \geqq 2$, *if and only if for all integers* $n \geqq 3$ *and all integers* q, $1 \leqq q \leqq (n-1)/2$, (2) *is true.*

Proof. If $n = 1$ or 2, then trivially, P_n does not satisfy (9), since there is no integer q in the range $1 \leqq q \leqq (n-1)/2$. Also, for all integers $x \geqq 2$, clearly

$$\pi(x + 2) \leqq \pi(x) + 1 = \pi(x) + \pi(2),$$

$$\pi(x + 3) \leqq \pi(x) + 2 = \pi(x) + \pi(3).$$

If $n \geqq 3$, then by Lemma IV, (1) is false for some pair of integers, $x \geqq 2$, $y \geqq 2$, if and only if there exists a prime P_n and an integer q, $1 \leqq q \leqq (n-1)/2$, such that $P_{n-q} + P_q + 1 \leqq P_n \leqq P_{n-q} + P_{q+1} - 3$. Hence (1) is true if and only if for all primes P_n, $n \geqq 3$, and all integers q, $1 \leqq q \leqq (n-1)/2$, either

$$P_n \geqq P_{n-q} + P_{q+1} - 1, \text{ or} \tag{10}$$

$$P_n \leqq P_{n-q} + P_q - 1 \text{ is true.} \tag{11}$$

But if $\pi(x+y) \leqq \pi(x) + \pi(y)$ for all integers x, $y \geqq 2$, (11) can never be true, since if it were true for some n and q, we would have

$$n = \pi(P_n) \leqq \pi(P_{n-q} + P_q - 1) \leqq \pi(P_{n-q}) + \pi(P_q - 1) = n - 1.$$

Hence, $\pi(x+y) \leqq \pi(x)+\pi(y)$ for all integers $x \geqq 2$, $y \geqq 2$, if and only if $P_n \geqq P_{n-q}+P_{q+1}-1$, for all $n \geqq 3$, and q, $1 \leqq q \leqq (n-1)/2$.

LEMMA V. *If for some $x \geqq 2$, $y \geqq 2$, (1) is false, then the smallest value of $x+y$ such that (1) is false, is prime.*

Proof. Let Z_0 be the smallest value of $x+y$ such that (1) is false. Let $Z_0 = X_0 + Y_0$, where X_0 and Y_0 are chosen such that

$$\pi(Z_0) = \pi(X_0 + Y_0) > \pi(X_0) + \pi(Y_0) \tag{12}$$

(such X_0 and Y_0 exist, since by hypothesis (1) is false for $X_0+Y_0=Z_0$).

Suppose $Z_0=X_0+Y_0$ were composite. We shall show that this assumption leads to a contradiction. Let P_w be the largest prime $<X_0+Y_0$. Let P_v be the largest prime $\leqq Y_0$. Define integers $r \geqq 1$ and $s \geqq 0$ by $P_w+r=X_0+Y_0$, and $P_v+s=Y_0$. Then $P_w-P_v+r-s=X_0$.

We have two cases.

Case I. If $r \geqq s$, then from (12) we have $\pi(X_0+Y_0)-\pi(Y_0)=\pi(P_w)-\pi(P_v) > \pi(X_0) = \pi(P_w - P_v + r - s) \geqq \pi(P_w - P_v)$. So $\pi(P_w) > \pi(P_w - P_v) + \pi(P_v)$. Therefore, (1) is false for $x=P_w-P_v$, $y=P_v$, $x+y=P_w<X_0+Y_0=Z_0$, contradicting the definition of Z_0.

Case II. If $r<s$, then define the integer $t \geqq 1$ by $r+t=s$. Then $X_0=P_w-P_v+r-s=P_w-P_v-t$, and $v \leqq \pi(P_v+t) \leqq \pi(P_v+s)=\pi(Y_0)=\pi(P_v)=v$. Hence, $\pi(Y_0) = \pi(P_v + t)$. Consequently, from (12), $\pi(X_0 + Y_0) = \pi(P_w) > \pi(X_0) + \pi(Y_0) = \pi(P_w-P_v-t)+\pi(P_v+t)$, and again we have a contradiction to the definition of Z_0. Hence, Z_0 is not composite, and so is prime.

THEOREM II. *If (1) is false for some integer $x+y$, then the smallest such value of $x+y$ is the smallest P_n for which (2) is false.*

Proof. If $\pi(x+y)>\pi(x)+\pi(y)$ for some integers $x \geqq 2$, $y \geqq 2$, then by Lemma V, the smallest value (if any) of $x+y$ such that this is true, is a prime, call it P_n. Suppose X_0 and Y_0 chosen so that $P_n=X_0+Y_0$, $Y_0>X_0$, and

$$n = \pi(P_n) = \pi(X_0 + Y_0) > \pi(X_0) + \pi(Y_0). \tag{13}$$

Clearly, $n \geqq 3$ (since $P_2=3$ and $P_1=2$ cannot be decomposed as the sum of two integers each $\geqq 2$). Hence we may choose q, $1 \leqq q \leqq n-2$, such that P_{n-q-1} is the largest prime $\leqq Y_0$ (and hence $Y_0 \leqq P_{n-q}-1$). Then from (13), $n>\pi(X_0) + \pi(Y_0)=\pi(X_0)+n-q-1$. Hence, $q \geqq \pi(X_0)$. Therefore,

$$P_{q+1} - 1 \geqq X_0 \tag{14}$$

(since if $X_0 \geqq P_{q+1}$, $\pi(X_0) \geqq q+1$), and so

$$P_n = X_0 + Y_0 \leqq P_{q+1} - 1 + Y_0 \leqq P_{q+1} + P_{n-q} - 2. \tag{15}$$

Furthermore, since $X_0<Y_0$, $X_0 \leqq P_{n-q}-1$. Hence, $X_0 \leqq \min(P_{n-q}-1, P_{q+1}-1)$,

$1 \leqq q \leqq n-2$. So in (14) we may take $q \leqq (n-1)/2$. And so from (15) there is a q, $1 \leqq q \leqq (n-1)/2$, such that P_n fails to satisfy (2).

The IBM 1620 computer installed at Wesleyan University(1) in Middletown, Connecticut, was programmed for inequality (2) by William Jeffreys, using D. N. Lehmer's tables of prime numbers, corrected so that $P_1 = 2$. The inequality (2) was found to hold for $n \leqq 9679$, $P_n \leqq 101{,}081$, which exhausted the capacity of the machine, which is of a card-punch type. In the programming, use was made of the fact that for $n \leqq 9679$, $P_n - P_{n-1} < 100$ and hence it was possible to represent each prime by two digits on the machine. Total running time was 19 hours. The use of a computer with a magnetic tape memory to examine still higher values of n would be of interest, since the programming of (2) would seem to be a relatively trivial problem.

Finally, it should be noted that the result of this paper does not replace, but only supplements, the results of Schinzel and Sierpiński. That is, we may now replace (C) by the statement: (1) is true if either x or $y \leqq 132$, or $x+y$ is $\leqq 101{,}081$.

References

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